A Decomposition Method for Nonconvex Quadratically Constrained Quadratic Programs

Chuangchuang Sun and Ran Dai

Abstract—This paper examines the nonconvex quadratically constrained quadratic programming (QCQP) problems using a decomposition method. It is well known that a QCQP can be transformed into a rank-one constrained optimization problem. Finding a rank-one matrix is computationally complicated, especially for large scale QCQPs. A decomposition method is applied to decompose the single rank-one constraint on original unknown matrix into multiple rank-one constraints on small scale submatrices. An iterative rank minimization (IRM) algorithm is then proposed to gradually approach all of the rank-one constraints. To satisfy each rank-one constraint in the decomposed formulation, linear matrix inequalities (LMIs) are introduced in IRM with local convergence analysis. The decomposition method reduces the overall computational cost by decreasing size of LMIs, especially when the problem is sparse. Simulation examples with comparative results obtained from an alternative method are presented to demonstrate advantages of the proposed method.

Index Terms—Quadratically Constrained Quadratic Programming; Rank Constraint Optimization; Semidefinite Programming; Matrix Decomposition; Sparse Matrix

I. INTRODUCTION

The general/nonconvex quadratically constrained quadratic programming (QCQP) problem has recently attracted significant interests due to its wide applications. For example, any polynomial problems to optimize a polynomial objective function while satisfying polynomial inequality constraints can be reformulated as general QCQP problems [1]. In addition, we can find QCQP applications in the areas of mixed integer quadratic programming [2]–[4], control allocation [5], support vector machines [6], [7], model predictive control [8], and optimal power flow [9], just to name a few.

Convexification and relaxation techniques have been commonly used when solving nonconvex optimization problems [10]–[12]. Efforts toward solving nonconvex QCQP problems have been pursued in two directions, obtaining a bound on the optimal value and finding a feasible solution. For simplicity, the QCQPs discussed below represent general/nonconvex QCQPs. Extensive relaxation methods, e.g., semidefinite [13] and linear relaxation [14], [15], have been investigated to obtain a bound on the optimal value of a QCQP. For example, the popularly used semidefinite relaxation method introduces a rank-one matrix to replace the quadratic objective function and constraints with linear matrices functions. The nonlinear rank-one constraint is substituted by a semidefinite constraint on the unknown matrix. The semidefinite relaxation finds a bound on the optimal value, even an exact solution in special cases [16]. However, finding a bound on the optimal value of QCQP does not imply generating an optimal solution, not even a feasible one in general. Detailed discussion of various relaxation approaches and the comparison of their relative accuracy is discussed in [17]. Methods searching for a feasible solution of QCQPs include iterative linearization, randomization, and branch and bound [1], [18].

Chuangchuang Sun and Ran Dai are with the Aerospace Engineering Department, Iowa State University, Ames, IA. Emails: ccsun@iastate.edu and dairan@iastate.edu

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When considering the rank-one constraint on an unknown matrix, a nonconvex QCQP can be equivalently formulated as a rank-constrained optimization problem (RCOP). Existing methods for solving RCOPs include alternating projection methods [19], [20] and combined linearization and factorization algorithms [21], [22]. However, these iterative approaches are sensitive to the initial guess. On the other facet, heuristic methods [23] such as the nuclear norm minimization, generally apply to rank minimization problems and are not applicable to RCOPs. Our previous work in [24]–[28] investigates an iterative method to gradually approach the rank-one constraint by introducing one additional linear matrix inequality (LMI) constraint to the semidefinite relaxation formulation. The newly introduced LMI increases the computation cost due to the additional linear constraints in order of $O(n^2)$ for a LMI in dimension $n$.

To improve scalability and computational efficiency when solving large scale QCQPs, a new formulation is proposed to decompose a large scale rank-one constraint into multiple ones in small scale. In addition, an iterative algorithm that drives all decomposed matrices to rank-one matrices is developed. Furthermore, we provide proof of convergence to a local optimum of the original QCQPs with at least a determined sublinear convergence rate for the proposed iterative algorithm. The decomposition formulation and associated iterative algorithm construct a new framework for solving nonconvex QCQPs with improved computational performance. The mixed-boolean problem in different scales is used to verify performance improvement in terms of computational time and objective value.

In the following, the nonconvex QCQP formulation and its equivalent transformation to an optimization problem with multiple rank constraints are introduced in §II. The iterative algorithm is discussed in §III with proof of convergence to a local optimum with at least a sublinear convergence rate. In §IV, simulation examples are presented to verify effectiveness of the proposed method. We conclude the paper with a few remarks in §V.

A. Preliminaries

Some notations used throughout this paper are introduced in this section. $\mathbf{x} \in \mathbb{R}^n$ denotes a vector in $n$-dimensional Euclidean space and $|\mathbf{x}|_1$ is the $1$-norm of $\mathbf{x}$. For a set $\alpha$, $|\alpha|$ denotes its cardinality. The set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$ and the set of $n \times n$ positive semidefinite (definite) matrices is denoted by $\mathbb{S}^n_+ (\mathbb{S}^n_{++})$. The notation $X \succeq 0$ ($X \succ 0$) means that the matrix $X \in \mathbb{S}^n$ is positive semidefinite (definite). ‘$I$’ represents an identity matrix in appropriate dimension. The trace of $X$ is denoted by $\text{tr}(X)$ and $\text{diag}(\bullet)$ denotes a matrix with ‘$\bullet$’ as the diagonal elements and zero for off-diagonal entries.

II. PROBLEM FORMULATION

A. General Homogeneous QCQPs

A general homogeneous QCQP problem can be expressed in the form of

$$J = \min_{\mathbf{x}} \mathbf{x}^T Q_0 \mathbf{x}$$

(2.1)
where $x \in \mathbb{R}^n$ is the unknown vector to be determined, $Q_j \in \mathbb{S}^n$, $j = 0, \ldots, m$, is an arbitrary symmetric matrix, and $c_j \in \mathbb{R}$, $j = 1, \ldots, m$. Since $Q_j$, $j = 0, \ldots, m$, are not necessarily positive semidefinite, problem in (2.1) is generally classified as nonconvex and NP-hard, requiring global search for its optimal solution. Any inhomogeneous QCQPs with linear terms can be reformulated as homogeneous ones by defining an extended vector $\tilde{x} = [x^T, t]^T$ as well as a new quadratic constraint $t^2 = 1$ [29]. For example, an inhomogeneous quadratic function can be transformed into an equivalent homogeneous one in form of

\[
x^T Q_j x + a_j^T x - c_j = [x^T \ t] \begin{bmatrix} Q_j & a_j/2 \\ a_j/2 & 0 \end{bmatrix} [x \ t] - c_j
\]

where $Q'_j = \begin{bmatrix} Q_j & a_j/2 \\ a_j/2 & 0 \end{bmatrix}$. Therefore, instead of solving inhomogeneous QCQPs, we can solve its equivalent homogeneous formulation expressed in (2.1). Without loss of generality, the following approach to nonconvex QCQP problems focuses on homogeneous QCQPs only.

B. Transformation of QCQPs into Rank-One Constrained Optimization Problems

By introducing a rank-one positive semidefinite matrix $X = x x^T$, problem in (2.1) can be rewritten as [13]

\[
J = \min_{X, x} \langle X, Q_0 \rangle
\]

s.t. \( \langle X, Q_j \rangle \leq c_j, \forall j = 1, \ldots, m, \) \hspace{1cm} (2.2)

\[
X = x x^T,
\]

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two matrices, i.e., $\langle A, B \rangle = \text{tr}(A^T B)$. However, the rank-one constraint on $X$ is nonlinear and it is computationally complicated to find the exact rank-one solution. One existing approach is to relax the rank-one constraint by a semidefinite constraint, denoted as $X \succeq x x^T$. The semidefinite relaxation approach generally yields a tighter lower bound on the optimal value of (2.1) than the one obtained from linearization relaxation technique [17]. However, the relaxation method will not generate an optimal solution of the unknown variables $x$, not even an infeasible solution in most cases.

The new approach introduced in this paper focuses on finding the unknown matrix satisfying rank-one constraint and the first step is to equivalently transform the rank-one constraint, $X = x x^T$, into multiple small scale quadratic equality constraints.

**Definition 1:** For $N = \{1, \ldots, n\}$, $\alpha_p \subseteq N$, $p = 1, 2, \ldots, P$, is a complete decomposition of $N$ if for any $1 \leq l_1, l_2 \leq n$, there exist $\alpha_p, p = 1, 2, \ldots, P$ such that $\{l_1, l_2\} \subseteq \alpha_p$.

As a result, problem in (2.2) can be equivalently transformed into

\[
J = \min_{X_{\alpha_p}, x_{\alpha_p}} \langle X, Q_0 \rangle
\]

s.t. \( \langle X, Q_j \rangle \leq c_j, \forall j = 1, \ldots, m, \) \hspace{1cm} (2.3)

\[
X_{\alpha_p} = x_{\alpha_p} x_{\alpha_p}^T, \quad p = 1, 2, \ldots, P,
\]

where $X_{\alpha_p}$ is a principle submatrix of $X$ consisting entries of $X$ with rows and columns indexed by $\alpha_p$ and $x_{\alpha_p}$ is a subset of $x$ with index set $\alpha_p$. Problem in (2.7) is equivalent to (2.2) if $\alpha_p \subseteq N$, $p = 1, 2, \ldots, P$, is a complete decomposition of $N$. To make it clear, a simple example of decomposition is demonstrated below.

**Example:** For $n = 3$, $\alpha_1 = \{1, 2\}$, $\alpha_2 = \{2, 3\}$ and $\alpha_3 = \{1, 3\}$ are a complete decomposition of $N = \{1, 2, 3\}$. When $n = 3$, problem in (2.2) is equivalent to

\[
J = \min_{X, x} \langle X, Q_0 \rangle
\]

s.t. \( \langle X, Q_j \rangle \leq c_j, \forall j = 1, \ldots, m, \)

\[
X_{\alpha_p} = x_{\alpha_p} x_{\alpha_p}^T, \quad p = 1, 2, \ldots, P, \hspace{1cm} (2.4)
\]

where $X = \begin{bmatrix} X_{\alpha_1} & X_{\alpha_2} & X_{\alpha_3} \end{bmatrix}$ and $x = [x_1, x_2, x_3]^T$.

Noteworthy, for sparse problems, i.e., problems with parameter matrices, $Q_j, \forall j = 0, \ldots, m$, being sparse, the complete decomposition will be less complex compared to the problems with dense parameter matrices. Figure 1 illustrates examples of the complete decomposition for problems with certain sparsity patterns. Mathematically, we define a matrix $A \in \mathbb{S}^n$ with $A_{ts} = 1$ if there exists a $j = 0, \ldots, m$ such that $(Q_j)_{ts} \neq 0$ and $A_{ts} = 0$ otherwise. Then (2.2) can be further simplified as

\[
J = \min_{X, x} \langle X, Q_0 \rangle
\]

s.t. \( \langle X, Q_j \rangle \leq c_j, \forall j = 1, \ldots, m \)

\[
X_{ts} = x_t x_t^T, \forall A_{ts} \neq 0. \hspace{1cm} (2.5)
\]

**Fig. 1.** Examples of complete decomposition for two problems with typical sparsity patterns: band (left) and block-arrow (right)

Through the above decomposition, the rank-one constraint on $X$ is equivalently transformed into multiple quadratic equality constraints on submatrices of $X$. However, when all decomposed matrices, $X_{\alpha_p}$, are rank-one matrices, it does not imply that $X_{\alpha_p} = x_{\alpha_p} x_{\alpha_p}^T, p = 1, 2, \ldots, P$, due to the coupled elements of $x$. For example, $X_{\alpha_1}$ obtained from the first and third submatrices in (2.4) are not necessarily equivalent to each other if we simply have rank-one constraint on both $X_{\alpha_1}$ and $X_{\alpha_3}$. To establish equivalent transformation, rank-one constraints are considered for extended submatrices such that

\[
\text{rank}( \begin{bmatrix} X_{\alpha_p} \\ x_{\alpha_p} \end{bmatrix}) = 1, \forall p = 1, 2, \ldots, P. \hspace{1cm} (2.6)
\]

Consequently, the original optimization problem with single rank-one constraint in (2.3) is converted into an equivalent optimization problem with multiple rank-one constraints on small size submatrices. The equivalent formulation is written as

\[
J = \min_{X, x} \langle X, Q_0 \rangle
\]

s.t. \( \langle X, Q_j \rangle \leq c_j, \forall j = 1, \ldots, m \)
III. AN ITERATIVE APPROACH FOR NONCONVEX QCQPs

A. Iterative Rank Minimization Approach

The new formulation of QCQP in (2.7) is to minimize a linear function under linear constraints and rank constraints, which is classified as a RCOP. Existing methods for solving RCOPs are subject to sensitive initial guess and uncertainty of convergence. An iterative approach with guaranteed convergence is proposed here to search for a local optimal solution that satisfies all of the decomposed rank-one constraints. In fact, for a rank-one matrix, the only nonzero eigenvalue is its largest one, which indicates the second largest eigenvalue of the rank-one matrix is zero. Therefore, a semidefinite constraint is introduced to put a upper bound on the second largest eigenvalue of the to-be-determined matrix.

Proposition 2: For a nonzero positive semidefinite matrix, \( U \in \mathbb{S}^n_+ \), its rank is no more than one if and only if there exists \( V \in \mathbb{R}^{n \times (n-1)} \), \( V^T V = I_{n-1} \) such that \( V^T X V = 0 \), where \( I_{n-1} \) is an identity matrix with dimension \( n-1 \).

Proof: Assuming the eigenvalues of \( U \) are sorted in increasing order, \( \lambda_1(U), \lambda_2(U), \ldots, \lambda_n(U) \), then the following inequality holds (Corollary 4.3.39 in [30]),

\[
\lambda_1^U + \lambda_2^U + \ldots + \lambda_{n-1}^U \leq \text{tr}(V^T UV)
\]

for any \( V \in \mathbb{R}^{n \times (n-1)} \) and \( V^T V = I_{n-1} \). When \( V \) are the eigenvectors corresponding to the \( n-1 \) smallest eigenvalues of \( U \), the equality holds for (3.8). For the sufficiency proof, when the rank of \( U \) is no more than one and \( V \) are the eigenvectors corresponding to the \( n-1 \) smallest eigenvalues of \( U \), \( V^T UV = 0_{(n-1) \times (n-1)} \), which indicates \( V^T X V = 0 \). To prove the necessity, we start from \( V^T UV = 0 \). It implies that \( 0 = \text{tr}(V^T UV) \geq \lambda_1(U) + \lambda_2(U) + \ldots + \lambda_{n-1}(U) \geq 0 \). Then it leads to \( \lambda_1(U) = \lambda_2(U) = \ldots = \lambda_{n-1}(U) = 0 \), which indicates rank of \( U \) is no more than one.

Based on the above discussion, we will substitute the rank one constraint, \( \text{rank} \left( \begin{bmatrix} X_{\alpha p} & X_{\alpha p} \\ X_{\alpha p}^T & 1 \end{bmatrix} \right) \), by the semidefinite constraint, \( r(p) I_{|\alpha p|} - (V^p)^T X(p) x(p) V^p \geq 0 \), where \( r(p) = 0 \), \( X(p) = X_{\alpha p} \) and \( x(p) = x_{\alpha p} \) are simplified notations, and \( V^p \in \mathbb{R}^{n \times (n+1) \times |\alpha p|} \) are the eigenvectors corresponding to the \( |\alpha p| \) smallest eigenvalues of \( X(p) \).

In the following, we denote \( \tilde{X}(p) \) for simplicity. However, before we solve \( \tilde{X}(p) \), we cannot obtain the exact \( V(p) \) matrix, thus an iterative rank minimization (IRM) method is proposed to gradually minimize the rank of \( \tilde{X}(p) \). At each step \( k \), we will solve the following semidefinite programming (SDP) problem formulated as

\[
J_k = \min_{X_k, r_k} \langle X_k, Q_k \rangle + w_k |r_k|_1
\]

s.t.

\[
\langle X_k, Q_j \rangle \leq c_j, \quad j = 1, \ldots, m
\]

\[
\tilde{X}_k \geq 0, \quad p = 1, 2, \ldots, P
\]

\[
r_k I_{|\alpha p|} - (V_k(p))^T \tilde{X}^k(p) V_k(p) \geq 0, \quad p = 1, 2, \ldots, P
\]

where \( w_k \) is an increasing weighting factor for \( |r_k|_1 \) with \( r_k = \{r_{(1)}^{(k)}, r_{(2)}^{(k)}, \ldots, r_{(P)}^{(k)} \} \) in \( k \)-th iteration and \( V_k(p) \) are the eigenvectors corresponding to the \( \alpha p \) smallest eigenvalues of \( \tilde{X}^k(p) \) solved at previous iteration \( k-1 \). In each iteration, we are trying to optimize the original objective function and at the same time minimize 1-norm of the newly introduced parameter \( r_k \) such that when \( r_k = 0 \), the rank one constraint on \( X_k(p) \), \( p = 1, 2, \ldots, P \), is satisfied. Meanwhile, since \( \tilde{X}^k(p) \) is constrained to be positive semidefinite, the term \( (V_k(p))^T \tilde{X}^k(p) V_k(p) \) is positive semidefinite as well, which implies that the value of \( r_k \) is nonnegative in order to satisfy \( r_k I_{\alpha p} - (V_k(p))^T \tilde{X}^k(p) V_k(p) \geq 0 \) in (3.9). The above approach is repeated until \( |r_k|_1 \leq \epsilon \), where \( \epsilon \) is a small threshold for stopping criteria. Once all of the submatrices \( \tilde{X}^k \) satisfy rank-one constraints, the original matrix \( X \) is rank one and the converged solution \( x \) satisfies \( X = xx^T \).

The IRM algorithm is summarized below.

**Algorithm 1: Iterative Rank Minimization**

**Input:** Parameters \( Q_0, Q_j, c_j, j = 1, \ldots, m, w_1, \alpha p, k_{\text{max}} \)

**Output:** Unknown rank one matrix \( X \) and unknown state vector \( x \)

**begin**

1) initialize Set \( k = 0 \), solve the SDP relaxation of (2.2) to find \( X_0(p) \) and obtain \( V_0(p) \) from \( X_0(p) \), set \( k = k + 1 \)

2) while \( |r_k|_1 > \epsilon \) \& \& \& \& \( k \leq k_{\text{max}} \)

3) Solve problem (3.9) and obtain \( X_k, x_k, r_k \)

4) Update \( V_k(p) \) from \( X_k(p) \), \( p = 1, 2, \ldots, P \)

5) \( k = k + 1 \), update \( w_{k+1} \) \& \& \& \& \( w_k \)

6) end while

7) Find \( X \) and \( x \)

**end**

Remark 3: For an SDP problem with \( m \) linear constraints and a linear matrix inequality of dimension \( n \), the computational time complexity is \( O(m(n^2 + m^3)) \) when solved by the interior point method [31]. Without decomposition, for original \( X \in \mathbb{S}^n \), the additional SDP constraint \( r_k I_{n-1} - V_k^T X_k V_k - 1 \geq 0 \) in (3.9) will introduce \( m' = O(n^2) \) additional linear constraints. Then the computational time complexity becomes \( O(n^6) \) and the corresponding memory is \( O(m^2) \), which is impractical for large scale QCQPs. Therefore, decomposing the original rank-one constraint into multiple one in small scale reduces size of each SDP constraints in (3.9).

Further analysis on sufficient conditions for local convergence of IRM and convergence rate is discussed below.

**B. Local Convergence of IRM Algorithm**

We first introduce a preparatory lemma.

**Lemma 4:** (Corollary 4.3.37 in [30]) For a given matrix \( X \in \mathbb{S}^n \) with eigenvalues in increasing order, denoted as \( \lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X) \), when \( m \leq n \), one has \( \lambda_{\text{max}}(X) \leq \lambda_{\text{max}}(X^T XV) \) for any \( V \in \mathbb{S}^m \), \( V^T V = I_m \).

Moreover, equality holds when columns of \( V \) are the eigenvectors of \( X \) associated with the \( m \) smallest eigenvalues.

**Assumption 5:** Problem in (2.1) is feasible and the relaxed problem in (2.2) is bounded. As a result, for each iteration \( k \) with optimum \( X_k \), there exists \( 0 < M < +\infty \) such that \( |\langle Q_0, X_k \rangle| \leq M \).

**Assumption 6:** There exists \( \delta_2 \geq 0 \) such that \( 0 \leq \delta_2 = J_{k+1} - J^* \leq M_j \), where \( J^* \) is a local optimum of (2.1) with \( |\langle r^* \rangle| = 0 \).

Moreover, there also exists a sufficiently small \( \delta_1 \geq 0 \) such that \( |\tilde{J}_k - J_{k+1}| \geq \delta_1 |r_k|_1 \).

**Proposition 7:** Assuming Assumptions 5 and 6 hold and the weighting factor satisfies \( 0 \leq w_{k+1} = w_k \leq \frac{\delta_1}{M_j} \) and \( \frac{\delta_1}{M_j} \leq w_{k+1} < +\infty \), then when \( X_k \) and \( r_k \) are sufficiently close to \( X^* \) and \( r^* = 0 \), respectively, \( |r_k|_1 \) converges to 0 with at least a sublinear convergence rate.
Proof: \( r_k \) introduced in (3.9) are slack variables and \( r_k^{(p)} \) obtained at each step \( k \) satisfies
\[
\begin{align*}
\frac{\partial^2}{\partial X^2} &= Q_0 + \sum_{p=1}^{m} \lambda_p Q_p + \sum_{p=1}^{m} T_p^T W_p \left( V^{(p)} S_1^{(p)} (V^{(p)})^T - S_2^{(p)} \right) W_p T_p = 0, \\
\frac{\partial^2}{\partial X^2} &= \sum_{p=1}^{m} T_p^T W_p \left( V^{(p)} S_1^{(p)} (V^{(p)})^T - S_2^{(p)} \right) E_p^T = 0, \\
\lambda_j (\lambda_j - c_j) = 0, \quad \lambda_j \geq 0, \quad \forall j = 1, \ldots, m, \\
S_2^{(p)} X^{(p)} = 0, \quad S_1^{(p)} (r^{(p)} I - (V^{(p)})^T X^{(p)} V^{(p)}) = 0, \\
X^{(p)} \geq 0, \quad r^{(p)} I - (V^{(p)})^T X^{(p)} V^{(p)} \geq 0, \\
S_1^{(p)} \geq 0, \quad S_2^{(p)} \geq 0, \quad \forall p = 1, \ldots, P.
\end{align*}
\]
By denoting
\[
F_1 = \sum_{p=1}^{P} \left( S_2^{(p)}, X^{(p)} \right),
\]
the above optimality conditions lead to
\[
\begin{align*}
F_1 &= \left\langle \sum_{p=1}^{P} T_p^T S_2^{(p)} T_p, \begin{bmatrix} X \\ T \end{bmatrix} \rightrangle \\
&= \left\langle \begin{bmatrix} \frac{\partial F_1}{\partial X} \\ \frac{\partial F_1}{\partial T} \end{bmatrix}, \begin{bmatrix} X \\ T \end{bmatrix} \rightrangle = 0. \quad (3.14)
\end{align*}
\]
As both \( \sum_{p=1}^{P} T_p^T S_2^{(p)} T_p \) and \( \begin{bmatrix} X \\ T \end{bmatrix} \) are positive semidefinite and \( F_1 = 0 \), we can get that the product of those two are zeros, so the principle submatrix
\[
\frac{\partial F_1}{\partial X} X + \frac{\partial F_1}{\partial T} T = 0. \quad (3.15)
\]
Similarly, by denoting
\[
F_2 = \sum_{p=1}^{P} \left( S_1^{(p)}, (V^{(p)})^T X^{(p)} V^{(p)} \right),
\]
we have
\[
\frac{\partial F_2}{\partial X} X + \frac{\partial F_2}{\partial T} T = 0. \quad (3.16)
\]
Moreover, since \( \frac{\partial F_1}{\partial X} + \frac{\partial F_2}{\partial X} = 0 \), we get
\[
\left( \sum_{p=1}^{P} T_p^T W_p \left( V^{(p)} S_1^{(p)} (V^{(p)})^T - S_2^{(p)} \right) W_p T_p \right) X = 0.
\]
Considering that \( X = xx^T \) and \( x \neq 0 \), then
\[
\sum_{p=1}^{P} T_p^T W_p \left( V^{(p)} S_1^{(p)} (V^{(p)})^T - S_2^{(p)} \right) W_p T_p x X = 0.
\]
The Lagrangian function for problem (2.3) is
\[
\mathcal{L}' = \langle Q_0, X \rangle + \sum_{j=1}^{m} \lambda_j \langle Q_j, X \rangle - c_j \]
\[
- \sum_{p=1}^{P} S_1^{(p)} \left( r^{(p)} I - (V^{(p)})^T X^{(p)} V^{(p)} \right) ,
\]
where the subscript \( k \) is omitted for simplicity, \( \lambda \in \mathbb{R}^m \), \( S_1^{(p)} \in \mathbb{R}_+^{m \times m} \), \( S_2^{(p)} \in \mathbb{S}_+^{m \times m} \), \( p = 1, \ldots, P \) are corresponding dual variables. At convergence point, the optimality conditions are expressed as
\[
\begin{align*}
\frac{\partial \mathcal{L}'}{\partial X} &= Q_0 + \sum_{j=1}^{m} \lambda_j Q_j + \sum_{p=1}^{P} M_p (T_p X T_p - T_{p} x x^T T_p), \\
\frac{\partial \mathcal{L}'}{\partial T} &= \sum_{p=1}^{P} T_p^T W_p \left( V^{(p)} S_1^{(p)} (V^{(p)})^T - S_2^{(p)} \right) W_p T_p = 0, \\
\frac{\partial \mathcal{L}'}{\partial \lambda} &= \sum_{j=1}^{m} \lambda_j (Q_j - c_j).
\end{align*}
\]
where \( \lambda' \in \mathbb{R}^m \) and \( M_p \in \mathbb{S}_+^{m \times m} \), \( p = 1, \ldots, P \) are corresponding dual variables. By setting \( \lambda'_j = \lambda_j \), \( \forall j = 1, \ldots, m \) and \( M_p = W_p^T (V^{(p)} S_1^{(p)} (V^{(p)})^T - S_2^{(p)} W_p \), \( \forall p = 1, \ldots, P \), it can be verified that \( (X, x, \lambda', M) \) satisfy all conditions listed below,
\[
Q_0 + \sum_{j=1}^{m} \lambda'_j Q_j + \sum_{p=1}^{P} T_p M_p T_p = 0,
\]
which are the KKT conditions of problem (2.3). Since problem in (2.3) is equivalent to the original problem (2.1), satisfying KKT conditions of problem (2.3) indicates that \((X^*, \mathbf{x}^*)\) is a local optimum pair of the original problem in (2.1).

IV. SIMULATION

To verify the feasibility and efficiency of the proposed decomposed formulation and associated IRM method, the mixed-boolean quadratic programming problems are considered here as simulation examples. Each iterative problem formulated in (3.9) is solved via the SDP solver, SeDuMi [32]. All of the simulation is run on a desktop computer with a 3.50 GHz processor and a 16 GB RAM.

The mixed-boolean quadratic programming problems are formulated as,

\[
J = \min_{\mathbf{x}} \mathbf{x}^T Q_0 \mathbf{x} \\
\text{s.t.} \quad \mathbf{x}^T Q_i \mathbf{x} + c_i \leq 0, \quad \forall i = 1, \ldots, m, \\
\mathbf{x}_j \in \{1, -1\}, \quad \forall j \in N_i,
\]

where \(\mathbf{x} \in \mathbb{R}^n\), \(m\) is the number of inequality constraints, \(N_i\) is the index set of the integer variables. The matrices \(Q_0\) and \(Q_i\), \(i = 1, \ldots, m\), are randomly generated and not necessarily positive semidefinite. Here the problem is sparse in the band pattern as defined before. Since the bivalent constraint on integer variables, \(x_j\), can be expressed as a quadratic equality constraint in the form of \((x_j + 1)(x_j - 1) = 0, \forall j \in N_j\), problem (4.17) can be converted to a nonconvex QCQP problem which can be solved by the proposed IRM method. The threshold for stopping criteria of IRM is set as 
\(\epsilon = 10^{-6}\).

We first set \(n = 60\) and compare the results from IRM to those obtained from a commercial mixed-integer nonlinear programming solver, ‘minlpBB’, which applies branch and bound method to search for optimal solutions [33]. Moreover, we compare the results obtained from the centralized formulation in (2.2) and two types of complete decomposition formulation, i.e., \(D_1: |\alpha_p| = \frac{n}{2}, P = 6\) and \(D_2: |\alpha_p| = \frac{n}{2}, P = 15\). Note that we assume the cardinality of each \(\alpha_p\) is the same for simplicity. 20 random cases are generated and solved via IRM, including three types of formulation, and a commercial solver, ‘minlpBB’. For each case, the objective value obtained from both methods are recorded in Fig. 2. After comparison, the objective value obtained from IRM, including three types of formulation is always smaller than the corresponding one computed from ‘minlpBB’ for all of the 20 cases. The average computational time of each iteration (seconds)/the average iteration number for those methods are ‘minlpBB’: 5.17/1, IRM/centralized: 98.92/10.75, IRM/D1: 41.16/10.05, and IRM/D2: 27.59/10.95.

In average, each iteration of \(D_2\) decomposition method reduces 73.11% of the computational time compared to the centralized one. It verifies that the decomposed formulation significantly reduces the computational time compared to the centralized formulation. Furthermore, the value of \(r^{(p)}\), representing the second largest eigenvalue of the unknown matrix \(X^{(p)}\), at each iteration is demonstrated in Fig. 3 for one selected case. As \(r^{(p)}\) converges to a number close to zero within a few iterations, Fig. 3 verifies the convergence of \(X^{(p)}\) in the IRM method to a rank one matrix. \(r^{(p)}\) for remaining cases also converge to zero and are not displayed here to save space.

Moreover, to demonstrate the reduction of computational time using the decomposition formulation for QCQPs at different scale,
This paper proposes a decomposition formulation and an iterative approach to solve nonconvex quadratically constrained quadratic programming (QCQP) problems. A general QCQP can be formulated as a rank-one constrained optimization problem. To improve scalability, a decomposition method is proposed to decompose the rank-one constraint into multiple ones in smaller scale. An iterative rank minimization (IRM) algorithm is then developed to drive all decomposed submatrices into rank-one matrices. The decomposition formulation makes the IRM algorithm more computationally efficient in solving large scale QCQPs. Theoretical analysis on convergence of the proposed method to a local optimum is provided. The effectiveness and improved performance of the new optimization framework is verified by comparative simulation results.

V. CONCLUSIONS

This paper proposes a decomposition formulation and an iterative approach to solve nonconvex quadratically constrained quadratic programming (QCQP) problems. A general QCQP can be formulated as a rank-one constrained optimization problem. To improve scalability, a decomposition method is proposed to decompose the rank-one constraint into multiple ones in smaller scale. An iterative rank minimization (IRM) algorithm is then developed to drive all decomposed submatrices into rank-one matrices. The decomposition formulation makes the IRM algorithm more computationally efficient in solving large scale QCQPs. Theoretical analysis on convergence of the proposed method to a local optimum is provided. The effectiveness and improved performance of the new optimization framework is verified by comparative simulation results.

REFERENCES