Convex Optimization for Energy-Efficient Traffic Control

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Abstract—This article presents a convex optimization approach to reduce fuel consumption of traffic flow on highways through speed limit control. By implementing Greenshields fundamental diagram, the solution to Moskowitz equations is expressed as linear equations with respect to vehicle inflow and outflow, which leads to generation of a linear traffic flow model. In addition, we build a quadratic function to estimate fuel consumption rate based on COPERT model. The energy-efficient traffic control problem is formulated as a convex quadratic optimization problem. Simulation results demonstrate significant reduction of fuel consumption, alleviation of congestion, and improved robustness using the proposed approach under high traffic demands.

Index Terms—Vehicle Fuel Consumption; Convex Optimization; Quadratic Programming; Macroscopic Traffic Flow Control

I. INTRODUCTION

Large scale complex transportation system is one of the indispensable infrastructures in our society. The dramatically increasing demands of transportation service leads to traffic congestion, energy wasting and pollution, as well as safety issues. To deal with these issues, intelligent traffic management strategies relying on advanced sensing and communication techniques are attracting researchers’ attention.

A fuel-efficient transportation system aims at reducing fuel consumption and emissions, e.g. CO, NO, CH₄ through eco-driving guidance. Recent work in areas of intelligent transpiration systems mostly focuses on reducing travel time [1], [2] or minimizing delay at signalized intersections [3], [4]. If fuel consumption is considered in evaluating the transportation system performance, it is necessary to analyze the effectiveness of current traffic control systems in terms of energy efficiency while guaranteeing the accomplishment of transportation tasks in desired time.

Existing energy-efficient traffic control strategies include adjusting signal time at intersections on urban roads [5], [6] and controlling the on-ramp metering rate at mainstream entry point of highways [7]–[9]. Most of existing eco-driving strategies have not included the dynamic traffic flow model which characterizes the evolution of traffic flow velocity and density [10], [11]. Although a second-order macroscopic traffic flow model, METANET, has been adopted in energy-efficient traffic management, it is time consuming to find a convergent solution when a nonlinear traffic flow model is considered [12]. Speed intervals have been used to obtain an approximate solution without solving nonlinear traffic flow dynamics, which results in accumulative error over time [9].

This work focuses on managing one type of highway infrastructure, dynamic speed limit signs, to control traffic speeds in order to reduce total fuel consumption during a specific time period while considering traffic flow dynamics. To generate a traffic control model that is computationally efficient and facilitates searching for an optimal control command, we aim at formulating the optimization performance index and the dynamic traffic flow model via convex functions. In this vein, the convex optimization approach will generate optimal speed profiles within polynomial computational time.

The macroscopic traffic flow model was first introduced by Lighthill and Whitham in the 1950s [13] and was intensively investigated afterwards. The fundamental traffic flow model is based on the continuous conservation law in the form of partial differential equations (PDE). Traditional discretization method used to solve Lighthill-Whitham-Richard (LWR) PDEs leads to a reduced model with low precision. Inspired by Barron-Jensen/Prankowski’s (B-J/P) solution for Hamilton-Jacobi (HJ) PDEs [14], we adopt B-J/P solution to Moskowitz HJ PDEs [13], [15], [16] to obtain more precise estimates on the traffic flow states. The solution can be explicitly expressed based on a fundamental diagram [17] associated with initial and boundary conditions. Furthermore, the solution to Moskowitz HJ PDEs can be simplified based on roadway decomposition and traffic status. Combining the simplified solution with the quadratic formulation of COPERT fuel consumption model [18], we formulate the energy-efficient traffic control problem as a convex quadratic optimization problem (CQOP).

The contribution of our work is: (1) finding of an explicit solution to Moskowitz HJ PDEs using a parabolic shaped fundamental diagram associated with initial and boundary conditions; and (2) constructing a macroscopic fuel consumption model and formulating the energy-efficient traffic control problem as a CQOP. In the following, we first introduce the problem and traffic flow model in II. The general solution of Moskowitz HJ PDEs and its simplified solution are described in III. IV presents the COPERT model and formulation of the energy-efficient traffic control problem. Simulation examples are demonstrated in V to verify efficiency and improved performance of the proposed method. We address the concluding remarks and summary in VI.

II. PROBLEM STATEMENT AND TRAFFIC FLOW DYNAMICS

A. Problem Statement

A one-dimensional, uniform highway section considered in this article is represented by [ξ, χ], where ξ and χ are upstream and downstream boundaries. We denote the vehicle density as ρ(t, x) per unit length for local position x ∈ [ξ, χ] at time t ∈ [0, tₘ]. The inflow and outflow are denoted as Qₓ and Qₓ, respectively. The vehicle velocity is a function of ρ and is denoted as v = v(ρ(t, x)).

The goal of the proposed traffic control strategy is to minimize the fuel consumption of vehicles on the concerned highway section for a desired time interval based on current traffic status by using controlling dynamic speed limit signals, shown as an example in Fig. 1.
B. Cauchy problem

The fundamental traffic flow model is based on the continuous conservation law in the form of PDE,
\[
\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0.
\]  

(2.1)

Introducing the cumulative vehicle count \(N(t, x)\), the vehicle density and flow can be calculated directly from the partial derivatives with respect to local position and time \(t\) in forms of
\[
\rho(t, x) = \frac{\partial N(t, x)}{\partial x}, \quad Q(t, x) = \frac{\partial N(t, x)}{\partial t}.
\]  

(2.2)

Substituting \(\rho(t, x)\) and \(Q(t, x)\) in (2.1) by (2.2) and then integrating both sides with respect to the local position generates the Moskowitz HJ PDE,
\[
\frac{\partial N(t, x)}{\partial t} - Q(\frac{\partial N(t, x)}{\partial x}) = 0.
\]  

(2.3)

Considering the initial, upstream, and downstream boundary conditions, i.e. \(c_{\text{ini}}(x)\), \(c_{\text{up}}(t)\), and \(c_{\text{down}}(t)\), together with the Moskowitz HJ PDE, the Cauchy problem [19] is formulated as,
\[
\begin{cases}
N(0, x) = c_{\text{ini}}(x) \\
N(t, \xi) = c_{\text{up}}(t) \\
N(t, \chi) = c_{\text{down}}(t).
\end{cases}
\]

(2.3)

The fuel efficient traffic control problem is to minimize the fuel consumption while satisfying the four equality constraints of the Cauchy problem listed above.

III. Explicit Solutions

A. B-J/F solution

We aggregate initial, upstream, and downstream boundary conditions in a value function condition, \(\epsilon(t, x)\), then the B-J/F solution to (2.3) can be represented by [20]
\[
N(t, x) = \inf_{(u, T) \in [w, v_f] \times \mathbb{R}_+} \left[ \epsilon(t - T, x - Tu) + TR(u) \right],
\]
where the convex transform \(R(u)\) is defined as
\[
R(u) = \sup_{\rho \in [0, \rho_f]} (Q(\rho) - up), \forall u \in [w, v_f],
\]
(3.4)

with \(w = \frac{\partial \rho}{\partial \rho} \big|_{\rho = \rho_0} < 0\), \(\rho_f > 0\) and \(v_f = \frac{\partial \rho}{\partial \rho} \big|_{\rho = 0} = 0\) denote the jam density and free-flow speed, respectively. However, the solution to HJ PDE may not be compatible under different conditions. Based on the Inf-morphism property [21] and Lax-Hopf formula 3.4, the last three equalities in the Cauchy problem can be converted into a set of inequalities.

Lemma 3.1: Compatibility Conditions [22]: The solution to HJ PDE is characterized by the Inf-morphism property, i.e. \(\epsilon(t, x) = \min_{i \in L} c_i(t, x)\), where \(L\) is the index number of value condition, the solution \(N(t, x) = \min_{i \in L} N_i(t, x)\) for \((t,x) \in [0, t_M]\times[\xi, \chi].\) The B-J/F solution to (2.3) satisfies the value conditions if and only if
\[
N_i(t, x) \geq c_j(t, x), \forall (t, x) \in \text{Dom}(c_i), (i,j) \in L^2
\]
(3.5)

Inequities in (3.5) represent the model constraints. By considering these constraints, we reduce B-J/F solutions to a subset representing the exact solution to the Cauchy problem. Solution to HJ PDE can be explicitly expressed based on Lax-Hopf formula. We integrate these expressions with piecewise affine value conditions to formulate model constraints. We define initial and boundary conditions as follows.

B. Piecewise affine initial and boundary conditions

We discretize time period \([0, t_M]\) and highway section \([\xi, \chi]\) into several small intervals using time step \(T\) and spatial step \(X.\)

The initial vehicle density \(\rho(0, x_k), k = 0, 1, 2, ... k_m,\) is assumed to be identical within segment \([x_k, x_{k+1}].\) Inflow and outflow remain constant during each time interval \([t_n, t_{n+1}]\) indexed by \(n = 0, 1, ... n_M.\) The initial and boundary conditions, \(c_{\text{ini}}, c_{\text{up}},\) and \(c_{\text{down}},\) can be decomposed into affine, locally-defined condition set, i.e. \(c_{\text{ini}}^k, c_{\text{up}}^n,\) and \(c_{\text{down}}^n.\) For example, the negative initial condition, \(-c_{\text{ini}}^k(t, x),\) represents the total number of vehicles at initial time contained between \([\xi, \chi].\) The upstream condition \(c_{\text{up}}^n(t, x)\) depicts total number of vehicles entering the roadway from initial to current time \(t.\) Hence we summarize the piecewise affine equations regarding initial and boundary conditions as [23],
\[
\begin{align*}
\left\{ \begin{array}{ll}
- \sum_{i=0}^{k-1} \rho(0, x_i)X - \rho(0, x_k)(x - kX), \\
\text{if } t = 0 & \forall x \in [x_k, x_{k+1}] \\
+ \infty, & \text{otherwise}
\end{array} \right. \\
& \quad \text{if } x \leq \xi \& t \in [t_n, t_{n+1}]
\end{align*}
\]
(3.6)

\[
\left\{ \begin{array}{ll}
\sum_{i=0}^{n-1} Q(t, \xi)T + Q(t_n, \xi)(t - nT), \\
\text{if } x = \xi \& t \in [t_n, t_{n+1}]
\end{array} \right. \\
& \quad \text{if } x \geq \chi \& t \in [t_n, t_{n+1}]
\]
(3.7)

\[
\left\{ \begin{array}{ll}
- \sum_{k=0}^{k_m} \rho(0, x_k)X, & \text{if } x = \chi \& t \in [t_n, t_{n+1}] \\
+ \infty, & \text{otherwise}
\end{array} \right.
\]
(3.8)

C. B-J/F explicit solution associated with initial and boundary conditions

The relationship between \(Q\) and \(\rho\) is represented by a fundamental diagram \(Q(\rho)\), which is established from empirical measurements. Here we consider the Greenshields fundamental diagram which is defined by
\[
Q(\rho) = -\frac{v_f}{\rho_f} \rho^2 + v_f \rho, \quad \rho \in [0, \rho_f],
\]
(3.9)

Given initial and boundary conditions, the B-J/F explicit solution based on the fundamental diagram is obtained by substituting the \(Q(\rho)\) and \(R(\rho)\) in (3.6)-(3.8) with (3.9) and (3.4), respectively. For initial condition, it includes two cases, uncongested initial case when \(0 \leq \rho(0, x) \leq \rho_c,\) where \(\rho_c\) is referred to as the critical density.
where \( Q'(\rho) = \frac{dQ(\rho)}{d\rho} |_{\rho = \rho(0,x)} \). For upstream boundary condition, corresponding explicit solution based on Lax-Hopf formula is

\[
N_{\text{up}}^n(t, x) = \begin{cases} 
\left( \frac{v_f - x - \frac{x}{v_f}}{4v_f} \right)^2 \rho_j \left(t - t_n + c_{\text{up}}^n(t_n, \xi)\right), & \text{if } t_n \leq t \leq t_n + T_0(\rho_{\text{up}}) \\
- \rho_{\text{up}}(x - \xi) + c_{\text{up}}^n(t, x), & \text{if } t_n + T_0(\rho_{\text{up}}) \leq t \leq t_{n+1} + T_0(\rho_{\text{up}}) \\
\left( \frac{v_f - x - \frac{x}{v_f}}{t - t_{n+1}} \right)^2 \rho_j \left(t - t_{n+1}\right)/4v_f + c_{\text{up}}^n(t_{n+1}, \xi), & \text{if } t \geq t_{n+1} + T_0(\rho_{\text{up}}).
\end{cases}
\] (3.12)

For downstream boundary condition, the corresponding explicit solution based on Lax-Hopf formula is

\[
N_{\text{down}}^n(t, x) = \begin{cases} 
\left( \frac{v_f - x - \frac{x}{v_f}}{4v_f} \right)^2 \rho_j \left(t - t_n + c_{\text{down}}^n(t_n, \chi)\right), & \text{if } t_n \leq t \leq t_n + T_0(\rho_{\text{down}}) \\
\rho_{\text{down}}(x - \chi) + c_{\text{down}}^n(t, x), & \text{if } t_n + T_0(\rho_{\text{down}}) \leq t \leq t_{n+1} + T_0(\rho_{\text{down}}) \\
\left( \frac{v_f - x - \frac{x}{t - t_{n+1}}}{t - t_{n+1}} \right)^2 \rho_j \left(t - t_{n+1}\right)/4v_f + c_{\text{down}}^n(t_{n+1}, \chi), & \text{if } t \geq t_{n+1} + T_0(\rho_{\text{down}})
\end{cases}
\] (3.13)

where \( T_0(\rho_{\text{up}}) = \frac{\pi - \epsilon}{Q'(\rho_{\text{up}})} \), \( T_0(\rho_{\text{down}}) = \frac{\pi - \epsilon}{Q'(\rho_{\text{down}})} \), \( \rho_{\text{up}} = \min\{\rho \in [0, \rho_i], Q(\rho) = Q(t, \xi)\} \), and \( \rho_{\text{down}} = \max\{\rho \in [0, \rho_i], Q(\rho) = Q(t, \chi)\} \). In the next subsection, we will discuss the simplified form of (3.10)-(3.13) for cases with relatively large inflow and outflow.

**D. Simplified B-J/F solution**

**Assumption 3.1:** A one lane highway with long distance can be decomposed into several segments with identical distance \( X \). B-J/F solution can be implemented in each segment.

**Assumption 3.2:** The highway section is required to handle cases with relatively large vehicle flow, i.e. flow at origin and ending is close to the road capacity \( Q_c \). In other word, one has \((1 - \frac{1}{v_f})\rho_c \leq \rho_{\text{up}} \leq \rho_c \) and \( \rho_c \leq \rho_{\text{down}} \leq (1 + \frac{1}{v_f})\rho_c \), where \( q \) is a user-specified parameter determining bounds of constraints. Substituting the above bounds on \( \rho_{\text{up}} \) and \( \rho_{\text{down}} \) in \( T_0(\rho_{\text{up}}) \) and \( T_0(\rho_{\text{down}}) \) yields \( T_0(\rho_{\text{up}}) \geq \sqrt{\frac{\pi - \epsilon}{v_f}} \) and \( T_0(\rho_{\text{down}}) \geq \sqrt{\frac{\pi - \epsilon}{v_f}} \).

Since each road segment is considered separately after decomposition, assumption 3.1 simply sets the initial density to be \( \{\rho(0,0), \rho(0,X)\} \) and denote \( \rho_{\text{ini}} = \rho(0,0) \) as vehicle density for each segment. To simplify the representation, \( \rho_{\text{ini}} \) is used for different segments in the following derivation. Furthermore, the plot of function (3.12) in Fig. 2 demonstrates that the slope of tangent line at each time instance increases when \( t \) varies from \( t_n + \frac{\pi - \epsilon}{v_f} \) to \( \frac{\pi - \epsilon}{v_f} \). Similar conclusion can be derived from the solution curve associated with the downstream boundary condition.

Assumption 3.2 introduces a linear approximation for (3.12) and (3.13) when \( t \geq \sqrt{\frac{\pi - \epsilon}{v_f}} \). Based on these discussion, the initial and boundary conditions for each road segment with modified notation \( Q_{\text{up}} = Q(t,0), Q_{\text{down}} = Q(t,X) \), and \( \rho_{\text{ini}} \), is expressed as

\[
c_{\text{ini}}^n(t, x) = \begin{cases} 
- \rho_{\text{ini}}x, & \text{if } t = 0 \& x \in [0, X] \\
\infty & \text{otherwise}
\end{cases}
\] (3.14)

With the updated initial and boundary conditions, the B-J/F solution associated with initial condition is simplified. For free-flow initial case with \( 0 \leq \rho_{\text{ini}} \leq \rho_c \), it reduces to

\[
N_{\text{up}}^n(t, x) = \left\{ \begin{array}{ll}
\frac{\rho_j x}{v_f} - \rho_{\text{ini}} t - \rho_{\text{ini}} x, & \text{if } 0 \leq t \leq \frac{x}{Q'(\rho_{\text{ini}})} \\
\frac{v_f}{4} \rho_j t + \frac{x^2}{2} \rho_j - \frac{x}{2} \rho_j, & \text{if } t \geq \frac{x}{Q'(\rho_{\text{ini}})}
\end{array} \right.
\] (3.17)

and for congested initial case with \( \rho_c \leq \rho_{\text{ini}} \leq \rho_j \), it becomes

\[
N_{\text{ini}}^n(t, x) = \left\{ \begin{array}{ll}
\frac{\rho_j x}{v_f} - \rho_{\text{ini}} t - \rho_{\text{ini}} x, & \text{if } 0 \leq t \leq \frac{x}{Q'(\rho_{\text{ini}})} \\
\frac{v_f}{4} \rho_j t + \frac{x^2}{2} \rho_j - \frac{x}{2} \rho_j, & \text{if } t \geq \frac{x}{Q'(\rho_{\text{ini}})}
\end{array} \right.
\] (3.18)

The solution components associated with boundary conditions become

\[
N_{\text{up}}^n(t, x) = \left\{ \begin{array}{ll}
\frac{\rho_j x}{v_f} - \rho_{\text{ini}} t - \rho_{\text{ini}} x, & \text{if } 0 \leq t \leq \frac{x}{Q'(\rho_{\text{ini}})} \\
\frac{v_f}{4} \rho_j t + \frac{x^2}{2} \rho_j - \frac{x}{2} \rho_j, & \text{if } t \geq \frac{x}{Q'(\rho_{\text{ini}})}
\end{array} \right.
\] (3.19)

\[
N_{\text{down}}^n(t, x) = \left\{ \begin{array}{ll}
\frac{\rho_j x}{v_f} - \rho_{\text{ini}} t - \rho_{\text{ini}} x, & \text{if } t \geq \frac{x}{Q'(\rho_{\text{ini}})}
\end{array} \right.
\] (3.20)
where \(a, c\) are slopes of the tangent line and \(b, f\) are the relative function values at \(t = t_n + \sqrt{\frac{X}{3.14}}\) and \(t = t_n + \sqrt{\frac{X}{3.15}}\).

### E. Model constraints

As described in §II, B-JF solution is the exact solution to Cauchy problem if inequality of (3.5) holds. We reduce these continuous inequalities for \(\forall (t, x) \in Dom(c_j)\) into a series of discrete inequalities by discretizing the continuous time interval into a set of small time intervals with step size \(T = 1\, \text{sec}\). By utilizing the linear interpolation on \([pT, (p+1)T]\), where \(p\) denotes the time index, the piecewise affine functions are built with respect to time \(t\). Therefore, the discrete inequality constraints are expressed in (3.21) in the next page.

Constraints (i) and (iii) in (3.21) are satisfied for \(x \in [\xi, \chi]\), \(t \in [0, t_m]\) in the simplified solution [20]. The remaining constraints in (3.21) are replaced by corresponding expressions defined in (3.14)-(3.20). For free-flow initial conditions with \(\rho_{ini} \leq \rho_c\) and discretized time index \(p \in [n, n_m]\) for \(t \in [pT, (p+1)T]\), (ii), (iv) in (3.21) becomes

\[(ii) \quad \frac{v_f}{\rho_f} t \geq Q_{up}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT) + \frac{1}{\rho_f} \rho_{ini} t \geq Q_{up}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT),\]

\[(iv) \quad \frac{v_f}{\rho_f} t \geq Q_{dp}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT) + \frac{1}{\rho_f} \rho_{ini} t \geq Q_{up}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT),\]

For congested initial conditions with \(\rho_{ini} \geq \rho_c\), constraints (ii) and (iv) in (3.21) become

\[(ii) \quad \frac{v_f}{\rho_f} t \geq Q_{up}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT) + \frac{1}{\rho_f} \rho_{ini} t \geq Q_{up}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT),\]

\[(iv) \quad \frac{v_f}{\rho_f} t \geq Q_{dp}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT) + \frac{1}{\rho_f} \rho_{ini} t \geq Q_{up}(t-pT) + \sum_{l=0}^{p-1} Q_{li}(t-pT),\]

For \(t_n \leq t \leq t_n + \sqrt{\frac{X}{3.14}}\), constraints (v) and (vi) in (3.21) become

\[(v) \quad \frac{v_f}{\rho_f} t \geq Q_{down}(t-nT) + (W + Q_{dp}(p-nT)) t - \sum_{l=0}^{p-1} Q_{down}(T) + X^2 \rho_f \geq 0,\]

\[(vi) \quad \frac{v_f}{\rho_f} t \geq Q_{down}(t-nT) + (W + Q_{dp}(p-nT)) t - \sum_{l=0}^{p-1} Q_{down}(T) + X^2 \rho_f \geq 0,\]

where \(W = \sum_{l=0}^{n-1} (Q_{li} - Q_{down} + (\rho_{ini} - \frac{\rho_f}{\rho_c}) X)\). For \(t \geq t_n + \sqrt{\frac{X}{3.15}}\), constraints (v) and (vi) in (3.21) become

\[(v) \quad (a - Q_{down}^p(t-nT) + Q_{down}^p(p-nT) + \rho_{ini} X + b - \sum_{l=0}^{p-1} Q_{li}^d T \geq 0 \quad \text{(3.26)},\]

\[(vi) \quad (c - Q_{down}^p(t-nT) + Q_{down}^p(p-nT) + f - \sum_{l=0}^{p-1} Q_{li}^d T \geq 0. \quad \text{(3.27)}\]

Equations (3.22)-(3.27) are model constraints describing traffic flow dynamics. Constraints (ii) in (3.22) and (iv) in (3.23) have been verified since \(\frac{v_f^2}{\rho_f^2} \geq \max_{c \in [0, t_m]} |Q_{up} - Q_{down}|\). Hence both of them are ignored in formulation of the following optimization problem.

### IV. FUEL CONSUMPTION MODEL AND FORMULATION OF OPTIMIZATION PROBLEM

COPERT model is a macroscopic model estimating the emission and fuel consumption based on average vehicle speed [12]. The quadratic form of emission or fuel consumption with respect to average speed \(v_a\) is expressed as

\[J = \sum_{k=1}^{n_m} X [c_0 v_a (t_n + 1, x_k) + c_1 v_a (t_n + 1, x_k) + c_2] \quad \text{(4.29)},\]

where \(c_0, c_1\) and \(c_2\) are parameters specified in terms of vehicle category, such as passenger cars, light duty vehicles, heavy duty vehicles, etc. We define average vehicle speed at time \(t\) and location \(x\) as \(v_a(t, x) = \{v_a(t, x) \in [0, v_f]\}\). From (3.9), one has \(v_a = \frac{v_f}{\rho_f} \rho_a(t, x) + v_f\), where \(\rho_a(t, x)\) is the average vehicle density. Assuming vehicles belong to class of EURO I, the speed range is \(13.1 - 130\, \text{km/h}\) and for each vehicle the cylinder capacity range is 1.4l - 2.0l, then the performance index based on fuel consumption rate in the unit of $/g/km$ is constructed as

\[J = \sum_{k=1}^{n_m} X \left[c_0 \rho_a(t_n + 1, x_k) + c_1 \rho_a(t_n + 1, x_k) + c_2\right] \quad \text{(4.29)},\]

where \(c_0 = 0.0144, c_1 = -2.314, c_2 = 135.44\). The quadratic form of the objective function is determined by \(Q_{up}(t_n, x_k)\) and \(Q_{down}(t_n, x_k)\). Moreover, there is no on-ramp or off-ramp considered in this problem. Thus the outflow of segment \(k\) is equivalent to the inflow of segment \(k + 1\) which leads to the additional equality constraints, \(Q_{down}(t_n, x_k) = Q_{up}(t_n, x_{k+1})\), \(k = 0, 1, \ldots, n_m - 1\). Combining the linear model constraints formulated in (3.24)-(3.27), the fuel efficient optimization problem is formulated by

\[\min. J = \text{(4.29)},\]

\[\text{s.t.} \quad A_{model} Y \leq b_{model}, \quad Q_{down} = Q_{up}(t_n, x_{k+1} - Q_{down}(t_n, x_k), \quad k = 0, 1, \ldots, n_m - 1\]

where \(A_{model}\) and \(b_{model}\) represent the parameter matrix and vector derived from the linear model constraints in (3.24)-(3.27). The unknown variable set, \(y = [Q_{down}^n, Q_{down}^{n+1}, \ldots, Q_{down}^{n+m}, Q_{up}^{n+1}, \ldots, Q_{up}^{n+m}]^T\), which includes inflow and outflow at time instant \(t_n\) for all segments. By solving the above COQP, we find the inflow and outflow for
each segment during \([t_n, t_{n+1}]\). From the determined \(Q(p_{n,k})\) and \(Q_{\text{down}}(p_{n,k})\) for \(k = 0, 1, \ldots, m\), the desired vehicle density for next time interval can be obtained from

\[
\rho(t_{n+1}, x_k) = \frac{(Q(p_{n,k}) - Q_{\text{down}}(p_{n,k}))T + \rho(t_n, x_k)X_k}{X_k}. \quad (4.30)
\]

To reach the desired vehicle density at next time instant \(t_{n+1}\), the desired speed of each segment at time interval \([t_n, t_{n+1}]\), i.e. \(v_d(t_n, x_k)\), is determined by

\[
v_d(t_n, x_k) = -\frac{v_f^2}{\rho_f^2} \rho(t_{n+1}, x_k) + v_f. \quad (4.31)
\]

V. SIMULATION EXAMPLES

This section applies the proposed optimization method for traffic control of an one lane highway section without on-ramps or off-ramps. Figure 1 illustrates the scenario of this one lane highway section which is divided into 12 segments with 1km in length for each one. The dynamic speed limit signs are displayed at the end of each segment and updated every one minute. The demand history at origin is specified in Fig. 3 and is updated before computing for the speed limit control in time next interval. If the demand is beyond the capacity, the inflow at origin is bounded \(Q_c\). Parameters related to Greenshields fundamental diagram are set as \(\rho_1 = 67\text{veh/km}, \rho_c = 33.5\text{veh/km},\) and \(v_f = 73.35\text{km/h}\). Two scenarios with different initial density are considered. The simulation period for both examples lasts for 2 hours. To make it simple, the driver compliance rate is assumed to be 100%.

A. Scenarios with No Congestion

In the first scenario, the initial density on every segment is assumed to be congestion free, i.e. \(\rho_{ini} = 25\text{veh/km}\) < \(\rho_c\). The optimization results generate time history of vehicle density for each segment in Fig. 5. To benchmark the improved performance of the optimization results, Fig. 4 demonstrates the vehicle density history of the same scenario without speed limit control. Comparison of the overall fuel consumption for the entire highway section during the two-hour interval with and without control are shown in Table I. It indicates that the total fuel consumption is significantly reduced under the optimal designed speed limit. Moreover, the proposed traffic control method makes the transportation system more robust to dramatic changes due to peak traffic demands. Therefore, the optimization approach will alleviate congestion under peak demand as a byproduct.

![Fig. 3. Demand at origin location](image3)

![Fig. 4. Density history without speed limit control for scenario 1](image4)

![Fig. 5. Density history with speed limit control for scenario 1](image5)

B. Scenario with Initial Congestion

The second scenario considers congestion on the last segment and the rest are assumed to be at the critical value of density capacity, i.e. \(\rho_{1-11} = 33.5\text{veh/km}\) and \(\rho_{12} = 40\text{veh/km}\). Vehicle demand at origin is consistent with scenario 1. Figures 6 and 7 demonstrate the density history without and with control, respectively. Apparently, in the case without speed limit control, the shock wave due to the initial congestion is propagated backwards, resulting in dramatic density increment until the demand drops down. In the case with optimal control, the shock wave is quickly damped out. Furthermore, we compare the total fuel consumption in Table I for the two-hour interval, which again demonstrates a substantial fuel reduction.

VI. CONCLUSION

This article proposes an efficient convex optimization method for minimizing fuel consumption of traffic flow modeled by Lighthill-Whitham-Richard partial differential equation. The explicit solution

\[
(i) N_{c_{up}}(0, x) \geq c_{ini}(0, x) \quad \forall(n, k) \in \{0, \ldots, n_m\} \times \{0, \ldots, k_m\} \quad \& \quad x \in [kX, (k + 1)X]
\]

\[
(ii) N_{c_{up}}(t, \xi) \geq c_{ini}(t, \xi) \quad \forall(k, p) \in \{0, \ldots, k_m\} \times \{0, \ldots, n_m\} \quad \& \quad t \in [pT, (p + 1)T]
\]

\[
(iii) N_{c_{down}}(0, x) \geq c_{ini}(0, x) \quad \forall(n, k) \in \{0, \ldots, n_m\} \times \{0, \ldots, k_m\} \quad \& \quad x \in [kX, (k + 1)X]
\]

\[
(iv) N_{c_{down}}(t, \xi) \geq c_{ini}(t, \xi) \quad \forall(k, p) \in \{0, \ldots, k_m\} \times \{0, \ldots, n_m\} \quad \& \quad t \in [pT, (p + 1)T]
\]

\[
(v) N_{c_{up}}(t, \chi) \geq c_{ini}(t, \chi) \quad \forall(n, p) \in \{0, \ldots, n_m\}^2 \quad \& \quad t \in [pT, (p + 1)T]
\]

\[
(vi) N_{c_{down}}(t, \chi) \geq c_{ini}(t, \chi) \quad \forall(n, p) \in \{0, \ldots, n_m\}^2 \quad \& \quad t \in [pT, (p + 1)T]
\]

(3.21)
to Cauchy problem is derived based on the Lax-Hopf formula. Linear model constraints to satisfy the initial and boundary conditions are considered in the Barron-Jensen/Frankowska solution. After modeling the performance index as a quadratic function, the fuel-efficient traffic control problem is formulated as a convex optimization problem. Simulation results demonstrate the reduced fuel consumption and improved robustness of traffic system. The feasibility of proposed optimization method is verified in both free-flow and initially congested scenarios.

REFERENCES


